Third Note on Weierstrass' Theory of Elliptic Functions.

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THE SIGMA-QUOTIENTS.

As long as the argument and the quasi-periods 2ω, $2\omega'$, remain the same, we may omit them, and write σ, σ_1, σ_2, σ_3, $\frac{\sigma}{\sigma_1}$, etc. The functions σ_1, σ_2, σ_3 are then thus defined,

$$\sigma_1 u = \frac{e^{-\eta u}\sigma(\omega+u)}{\sigma\omega} = \frac{e^{\eta u}\sigma(\omega-u)}{\sigma\omega},$$

$$\sigma_2 u = \frac{e^{-\eta'' u}\sigma(\omega''+u)}{\sigma\omega''} = \frac{e^{\eta'' u}\sigma(\omega''-u)}{\sigma\omega''},$$

$$\sigma_3 u = \frac{e^{-\eta' u}\sigma(\omega'+u)}{\sigma\omega'} = \frac{e^{\eta' u}\sigma(\omega'-u)}{\sigma\omega'},$$

which are seen to be even functions. These apparently arbitrary definitions flow naturally from considerations connected with the "pocket edition,"

$$\wp u - \wp v = \frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v}.$$

Since $\wp\omega = e_1$, $\wp(\omega+\omega') = e_2$, $\wp\omega' = e_3$, we have, using the general mark α, and writing $v = \omega_\alpha$

$$\wp u - e_\alpha = \frac{\sigma(u+\omega_\alpha)\,\sigma(u-\omega_\alpha)}{\sigma^2 u . \sigma^2 \omega_\alpha}.$$

But $\wp u$ is a truly periodic function: it remains therefore to examine the periodicity of σu. In the second note, p. 261, I have shown that

$$\sigma u = u \Pi'_w \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}}$$

$$w = m . 2\omega + m' 2\omega'$$

degenerates into the sine when $m' = 0$, or

$$\lim\,(\sigma u)_{m'=0} = \frac{2\omega}{\pi}\, e^{\frac{1}{6}\left(\frac{u\pi}{2\omega}\right)^2} . \sin\frac{u\pi}{2\omega}.$$

It was also shown that

$$\frac{\sin \pi x}{\pi} = x \prod_{n=-\infty}^{+\infty}{}' \left[\left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}\right] = s(x).$$

The following definitions are introduced as convenient:

$$\frac{d}{dx} \log s(x) = \frac{s'(x)}{s(x)} = s_1 x = \frac{1}{x} + \sum_{n=-\infty}^{+\infty}{}' \left(\frac{1}{x-n} + \frac{1}{n}\right)$$

$$-\frac{d}{dx} \cdot s_1 x = s_2 x = \sum_{n=-\infty}^{+\infty} \frac{1}{(x-n)^2}$$

in which last series the value $n = 0$ is included. One sees that $s_2(x+1) = s_2(x)$, and $s_2 x$ is periodic. Incidentally it may be remarked that on comparing the developments for $\sin \pi x$ and $s(x)$,

$$\sin \pi x = \pi x - \frac{\pi^3 x^3}{3!} + \frac{\pi^5 x^5}{5!} - \dots$$

$$\frac{\sin \pi x}{\pi} = s(x) = x\left(1 - \left(\frac{x}{1}\right)^2\right)\left(1 - \left(\frac{x}{2}\right)^2\right)\left(1 - \left(\frac{x}{3}\right)^2\right)\dots$$

$$= x - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right)x^3$$

$$+ \left(\frac{1}{2^2} + \frac{1}{3^2} + \dots\right)x^5$$

$$- \dots\dots\dots\dots$$

or, $$s(x) = x - x^3 \sum_{1}^{+\infty} \frac{1}{n^2} + x^5 \sum_{m=1}^{+\infty} \sum_{n=2}^{+\infty} \frac{1}{m^2 n^2} - x^7 \sum_{m=1}^{\infty} \sum_{n=2}^{\infty} \sum_{p=3}^{\infty} \frac{1}{m^2 . n^2 . p^2} + \dots$$

$$m < n \qquad\qquad m < n < p$$

whence $$\Sigma \frac{1}{n^2} = \frac{\pi^2}{3!}, \quad \Sigma\Sigma \frac{1}{m^2 n^2} = \frac{\pi^4}{5!}, \text{ etc.}$$

We can now in $$\sigma u = u \Pi' \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}},$$

give to m' a constant value and carry out the multiplication with respect to m. For $m' = 0$, we have

$$\sigma u = u \Pi' \left(1 - \frac{u}{m 2\omega}\right) e^{\frac{u}{m2\omega} + \frac{1}{2}\left(\frac{u}{m2\omega}\right)^2}$$

$$= u \Pi' \left(1 - \frac{u}{m 2\omega}\right) e^{\frac{u}{m2\omega}} . \Pi e^{\frac{1}{2}\left(\frac{u}{m2\omega}\right)^2}.$$

But $$\Pi e^{\frac{1}{8} \cdot \frac{1}{m^2} \cdot \frac{u^2}{\omega^2}} = e^{\frac{1}{8} \frac{\pi^2}{6} \cdot \frac{u^2}{\omega^2}},$$

and $$u \Pi' \left[\left(1 - \frac{u}{m 2\omega}\right) e^{\frac{u}{m2\omega}}\right] = \frac{2\omega}{\pi} \sin \frac{u\pi}{2\omega},$$

which two factors furnish that part of σu corresponding to those values of m and m' represented in the plane of complex number by points on the real axis distant from each other by 2ω; in other words to the numbers

$$0,\ \pm 1.2\omega,\ \pm 2.2\omega,\ \pm 3.2\omega, \ldots$$

Employing again
$$su = u\Pi'\left(1-\frac{u}{n}\right)e^{\frac{u}{n}},$$
we have
$$\frac{s(u-a)}{s(-a)} = \frac{u-a}{-a}\Pi'\left(1-\frac{u}{n+a}\right)e^{\frac{u}{n}}.$$
Remarking now the identity
$$\frac{u}{n} = \frac{u}{n+a} + \frac{u}{n} + \frac{u}{-n-a}$$
and also
$$\frac{s'(-a)}{s(-a)} = -\frac{1}{a} + \Sigma'\left(\frac{1}{-n-a}+\frac{1}{n}\right)$$
there appears
$$\frac{s(u-a)}{s(-a)} = \left(1-\frac{u}{a}\right)\Pi'\left(1-\frac{u}{n+a}\right)e^{\frac{u}{n+a}+\frac{u}{n}+\frac{u}{-n-a}}.$$
But
$$\frac{s'(-a)}{s(-a)} = s_1(-a)$$
therefore
$$\frac{s(u-a)}{s(-a)} = \left(1-\frac{u}{a}\right)\Pi'\left[\left(1-\frac{u}{n+a}\right)e^{\frac{u}{n+a}}\right].e^{us_1(-a)+\frac{u}{a}}$$
or, taking up $\left(1-\frac{u}{a}\right)e^{\frac{u}{a}}$ into the product as the value of $\left(1-\frac{u}{n+a}\right)e^{\frac{u}{n+a}}$ for $n=0$, we can drop the accent of the product sign and write
$$\frac{s(u-a)}{s(-a)} = \Pi\left(1-\frac{u}{n+a}\right)e^{\frac{u}{n+a}}.e^{us_1(-a)},$$
whereby the only restriction as to a is that it must not be an integer. Transposing,
$$\Pi\left(1-\frac{u}{n+a}\right)e^{\frac{u}{n+a}} = \frac{s(u-a)}{s(-a)}.e^{-u.s_1(-a)}.$$

We are now ready to decompose the sigma-product
$$\Pi\left(1-\frac{u}{m.2\omega+m'2\omega'}\right)e^{\frac{u}{m2\omega+m'2\omega'}+\frac{1}{2}\left(\frac{u}{m2\omega+m'2\omega'}\right)^2},$$
$$m = 0,\ \pm 1,\ \pm 2, \ldots \pm\infty,$$
$$m' = \quad \pm 1,\ \pm 2, \quad \pm\infty,$$
where $m'=0$ is omitted, as already accounted for, and consequently the accent on the product sign is dropped. Dividing by $2\omega_1$, the formula becomes
$$\Pi\left(1-\frac{\frac{u}{2\omega}}{m+m'\frac{\omega'}{\omega}}\right)e^{\frac{\frac{u}{2\omega}}{m+m'\frac{\omega'}{\omega}}}.e^{\frac{1}{2}\left(\frac{\frac{u}{2\omega}}{m+m'\frac{\omega'}{\omega}}\right)^2}.$$

Writing as before
$$s_2(x) = -\frac{d}{dx}s_1 x = \Sigma \frac{1}{(n-x)^2},$$
the second exponential factor becomes
$$e^{\frac{1}{2}\left(\frac{u}{2\omega}\right)^2 \cdot \Sigma \frac{1}{\left(m+m'\frac{\omega'}{\omega}\right)^2}} = e^{\frac{1}{2}\left(\frac{u}{2\omega}\right)^2 \cdot s_2\left(-m'\frac{\omega'}{\omega}\right)}$$
for each particular value of m', the summation being taken with respect to m alone. The rest of the product is
$$\Pi\left(1-\frac{\frac{u}{2\omega}}{m+m'\frac{\omega'}{\omega}}\right)e^{\frac{\frac{u}{2\omega}}{m+m'\frac{\omega'}{\omega}}} = \frac{s\left(\frac{u}{2\omega}-m'\frac{\omega'}{\omega}\right)}{s\left(-m'\frac{\omega'}{\omega}\right)}.e^{-\frac{u}{2\omega}\cdot s_1\left(-m'\frac{\omega'}{\omega}\right)}$$
by the formula above deduced. Collecting the four factors, we have
$$\sigma u = e^{\frac{1}{8}\cdot\frac{\pi^2}{6}\cdot\frac{u^2}{\omega^2}} s\left(\frac{u}{2\omega}\right).\prod_{m'=-\infty}^{+\infty}\left[e^{\frac{1}{2}\left(\frac{u}{2\omega}\right)^2 s_2\left(-m'\frac{\omega'}{\omega}\right)}.e^{\frac{-u}{2\omega}\cdot s_1\left(-m'\frac{\omega'}{\omega}\right)}.\frac{s\left(\frac{u}{2\omega}-m'\frac{\omega'}{\omega}\right)}{s\left(-m'\frac{\omega'}{\omega}\right)}\right].$$
This formula can however be simplified in form by multiplying together the factors in pairs and taking the product from 1 to ∞, instead of from $-\infty$ to $+\infty$. For we had
$$s(x) = x\prod_{-\infty}^{+\infty}{}'\left(1-\frac{x}{n}\right)e^{\frac{x}{n}},$$
$$s_1(-x) = -\frac{1}{x}+\sum_{n=-\infty}^{+\infty}{}'\left(\frac{1}{-x-n}+\frac{1}{n}\right),$$
$$s_2(-x) = \sum_{-\infty}^{+\infty}\frac{1}{(x+n)^2},$$
whence it appears that $s_1(-x) = \sum\limits_{n=1}^{+\infty}\frac{-2x}{x^2-n^2}$, or is an odd function, consequently
$$\sum_{m'=-\infty}^{+\infty} s_1(-m') = \sum_{m'=1}^{+\infty} s_1(-m') + \sum_{m'=1}^{+\infty} s_1(+m') = 0,$$
and one exponential factor disappears, and the formula now reads
$$\sigma u = 2\omega e^{\frac{1}{8}\cdot\frac{\pi^2}{6}\cdot\frac{u^2}{\omega^2}} s\frac{u}{2\omega}.e^{\frac{1}{2}\left(\frac{u}{2\omega}\right)^2\Sigma\left[s_2\left(-m'\frac{\omega'}{\omega}\right)+s_2\left(+m'\frac{\omega'}{\omega}\right)\right]}\prod_{m'=1}^{+\infty}\frac{s\left(\frac{u}{2\omega}-m'\frac{\omega'}{\omega}\right)s\left(\frac{u}{2\omega}+m'\frac{\omega'}{\omega}\right)}{s\left(-m'\frac{\omega'}{\omega}\right).s\left(m'\frac{\omega'}{\omega}\right)}$$
$$= 2\omega e^{\frac{1}{8}\frac{\pi^2}{6}\frac{u^2}{\omega^2}} s\frac{u}{2\omega}.e^{\frac{1}{8}\frac{\pi^2}{6}\frac{u^2}{\omega^2}+\left(\frac{u}{2\omega}\right)^2\sum\limits_{m'=1}^{+\infty}s_2\left(m'\frac{\omega'}{\omega}\right)}\prod_{m'=1}^{+\infty}\frac{s\left(\frac{u}{2\omega}-m'\frac{\omega'}{\omega}\right)s\left(\frac{u}{2\omega}+m'\frac{\omega'}{\omega}\right)}{s\left(-m'\frac{\omega'}{\omega}\right)s\left(m'\frac{\omega'}{\omega}\right)}$$
$$= 2\omega s\frac{u}{2\omega}.e^{\left(\frac{u}{2\omega}\right)^2\left[\frac{1}{2}\frac{\pi^2}{6}+\Sigma s_2\left(m'\frac{\omega'}{\omega}\right)\right]}\Pi\frac{s\left(\frac{u}{2\omega}-m'\frac{\omega'}{\omega}\right)s\left(\frac{u}{2\omega}+m'\frac{\omega'}{\omega}\right)}{s\left(-m'\frac{\omega'}{\omega}\right)s\left(m'\frac{\omega'}{\omega}\right)}$$

and, on passing from the s and s_2 to the sine,

$$\sigma u = \frac{2\omega}{\pi} \sin \frac{u\pi}{2\omega} \cdot e^{\frac{1}{6}\left(\frac{\pi u}{2\omega}\right)^2} \Pi \frac{\sin \frac{\pi}{2\omega}(2m'\omega' - u) \sin \frac{\pi}{2\omega}(2m'\omega' + u)}{\sin^2 \frac{m'\omega'\pi}{\omega}} \cdot e^{\frac{\left(\frac{u\pi}{2\omega}\right)^2}{\sin^2 \frac{m'\omega'\pi}{\omega}}}.$$

Professor Schwarz writes

$$\eta = \frac{\pi^2}{2\omega}\left\{\frac{1}{6} + \sum_{m'} \frac{1}{\sin \frac{m'\omega'\pi}{\omega}}\right\},$$

whereupon the sigma-function is thus represented as a singly infinite product of sines

$$\sigma u = e^{\frac{\eta u^2}{2\omega}} \cdot \frac{2\omega}{\pi} \cdot \sin \frac{u\pi}{2\omega} \prod_{m'} \left(1 - \frac{\sin^2\left(\frac{u\pi}{2\omega}\right)}{\sin^2\left(\frac{m'\omega'\pi}{\omega}\right)}\right).$$

On substituting $u + 2\omega$ for u the expression becomes

$$\sigma(u + 2\omega) = -e^{2\eta(u+\omega)} \sigma u,$$

from which by logarithmic differentiation and writing $u = -\omega$, we find

$$\eta = \frac{\sigma'\omega}{\sigma\omega},$$

Recurring now to the pocket edition

$$\wp u - e_\alpha = \frac{\sigma(u + \omega_\alpha)\sigma(u - \omega_\alpha)}{\sigma^2 u \sigma^2 \omega_\alpha}.$$

From the definitions at the beginning of this paper

$$\sigma_\alpha^2 u = e^{-2\eta_\alpha u} \frac{\sigma^2(\omega_\alpha - u)}{\sigma^2 \omega_\alpha},$$

so that

$$\wp u - e_\alpha = \left(\frac{\sigma_\alpha u}{\sigma u}\right)^2;$$

which is the simplest form of a doubly periodic function. In the second note was deduced the equation $(\wp' u)^2 = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3)$, and on comparison with the above

$$\wp' u = 2 \frac{\sigma_1 u . \sigma_2 u . \sigma_3 u}{\sigma u . \sigma u . \sigma u}.$$

The sigma-quotients have not the same pair of fundamental periods as the sigma-function itself. But while

σu	has the quasi-periods	$2\omega,\ 2\omega'$
$\frac{\sigma_1 u}{\sigma u}$	has the periods	$2\omega,\ 4\omega'$
$\frac{\sigma_3 u}{\sigma u}$	" "	$4\omega,\ 2\omega'$
$\frac{\sigma_2 u}{\sigma u}$	" "	$4\omega,\ 4\omega'$.

This is shown in the following manner. It will be noticed that aside from exponential and constant factors, the function σ_1, σ_2, σ_3, are formed from σu by increasing the argument u by the half-periods ω, $\omega'' = \omega + \omega'$, ω', respectively. If we write $\bar{w} = r\omega + r'\omega'$, $\bar{\eta} = r\eta + r'\eta'$ instead of $w = m2\omega + m'2\omega'$, it is evident that w and $\bar{w}$ will only then be equivalent when both r and r' are even. We have then

$$\sigma(u + 2\bar{w}) = \varepsilon\sigma u . e^{2\bar{\eta}(u+\bar{w})},$$

$$\sigma(u + w) = \varepsilon\sigma(u - w)e^{2\bar{\eta}u} = -\varepsilon\sigma(w - u)e^{2\bar{\eta}u},$$

and, writing $u = 0$, $\sigma\bar{w} = -\varepsilon\sigma(w)$, or $\varepsilon = -1$ when either r or r' is odd. If both are even, then $\bar{w} \equiv w$ and $\sigma w = 0$. To determine the value of ε in this case, develop both sides according to powers of u

$$u . \sigma'\bar{w} + u^2 + \ldots = \varepsilon . u . \sigma'\bar{w} + \ldots$$

and $\varepsilon = +1$ when both r and r' are even. Now the formula

$$(r + 1)(r' + 1) - 1 = rr' + r + r'$$

is only even when both r and r' are even; we can write therefore,

$$\sigma(u + 2\bar{w}) = (-1)^{rr'+r+r'} . \sigma u . e^{2\bar{\eta}(u+\bar{w})}$$

$$\sigma(u + \omega_\alpha + 2\bar{w}) = (-1)^{rr'+r+r'} . \sigma(u + \omega_\alpha)e^{2\bar{\eta}(u+\omega_\alpha+\bar{w})};$$

but, from the definition $\sigma(u + \omega_\alpha) = e^{\eta_\alpha u} . \sigma_\alpha u . \sigma\omega_\alpha$,

whence, writing for u, $u + 2\bar{w}$

$$\sigma(u + \omega_\alpha + 2\bar{w}) = e^{\eta_\alpha(u+2\bar{w})}\sigma_\alpha(u + 2\bar{w}) . \sigma\omega_\alpha,$$

and, equating the right-hand members,

$$\sigma_\alpha(u + 2\bar{w})\sigma\omega_\alpha = (-1)^{rr'+r+r'} . \sigma(u + \omega_\alpha)e^{2\bar{\eta}(u+\omega_\alpha+\bar{w}) - \eta_\alpha(u+2\bar{w})}$$

or, writing $u - \bar{w}$ for u

$$\sigma_\alpha(u + \bar{w})\sigma\omega_\alpha = (-1)^{rr'+r+r'} . \sigma_\alpha(u - \bar{w})e^{2(\bar{\eta}\omega_\alpha - \eta_\alpha\bar{w}) + 2\bar{\eta}u},$$

$$\frac{\sigma_\alpha(u + \bar{w})}{\sigma_\alpha(u - \bar{w})} = (-1)^{rr'+r+r'} e^{2(\bar{\eta}\omega_\alpha - \eta_\alpha\bar{w}) + 2\bar{\eta}u},$$

and for $u = 0$, since $\sigma_\alpha(-\bar{w}) = \sigma_\alpha(+\bar{w})$ we have for the determination of r and r',

$$1 = e^{2(\bar{\eta}w_\alpha - \eta_\alpha\bar{w}) + (rr'+r+r')\pi i}.$$

For the case $\alpha = 1$, we shall have $\bar{\eta} = r\eta + r'\eta'$, $\bar{w} = r\omega + r'\omega'$, $\eta_\alpha = \eta$, $\omega_\alpha = \omega$, and

$$1 = e^{2r'(\eta'\omega - \eta\omega') + (rr'+r+r')\pi i}$$

But

$$\eta'\omega - \eta\omega' = \pm\frac{\pi i}{2},$$

whence

$$\sigma_1(u + 2\bar{w}) = (-1)^{rr'+r} . \sigma_1 u . e^{2\bar{\eta}}(u + \bar{w}).$$

For the case $\alpha = 3$, we shall have

$$\eta_\alpha = \eta', \ \omega_\alpha = \omega', \ 2(\bar{\eta}\omega_\alpha - \eta_\alpha\bar{w}) = 2r(\eta\omega' - \eta'\omega),$$

and $\sigma_3(u+2\bar{w}) = (-1)^{rr'+r'}\sigma_3 u \,.\, e^{2\bar{\eta}(u+\bar{w})}$,

and likewise $\sigma_2(u+2\bar{w}) = (-1)^{rr'}\sigma_2 u e^{2\eta(u+\bar{w})}$;

so that

$$\frac{\sigma_1(u+2\bar{w})}{\sigma(u+2\bar{w})} = -(-1)^{rr'+r}\cdot\frac{\sigma_1 u}{\sigma u},$$

$$\frac{\sigma_3(u+2\bar{w})}{\sigma(u+2\bar{w})} = -(-1)^{rr'+r'}\cdot\frac{\sigma_3 u}{\sigma u},$$

$$\frac{\sigma_2(u+2\bar{w})}{\sigma(u+2\bar{w})} = -(-1)^{rr'}\cdot\frac{\sigma_2 u}{\sigma u}.$$

In order therefore that $2w = 2(r\omega + r'\omega')$ may be a period of $\frac{\sigma_1 u}{\sigma u}$, we must have $(-1)^{rr'+r+1} = 1$, or $rr' + r + 1 =$ even, $r(r'+1) =$ odd, that is $r =$ odd, $r' =$ even, so that $2\bar{w} = 2m\omega + 4m'\omega'$, where m and m' are integers. In like manner, for $\alpha = 3$ we shall have $2\bar{w} = 4m\omega + 2m'\omega'$, and for $\alpha = 2$, $2\bar{w} = 4m\omega + 4m'\omega'$.

The relation will now be shown between the sigma-quotients on the one hand, and the notation of Jacobi and Abel on the other. The Jacobian differential equation is

$$\left(\frac{dx}{du}\right)^2 = (1-x^2)(1-k^2x^2).$$

In the second note, p. 267, we had

$$(\wp' u)^2 = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3),$$

or, since

$$\wp u - e_\lambda = \left(\frac{\sigma_\lambda u}{\sigma u}\right)^2;\ \lambda = 1,\ 2,\ 3,$$

$$\wp' u = -2\,\frac{\sigma_\lambda u\,.\,\sigma_\mu u\,.\,\sigma_\nu u}{\sigma u\,.\,\sigma u\,.\,\sigma u}.$$

Writing now for convenience $\frac{\sigma u}{\sigma_\lambda u} = \xi_{0\lambda}$, $\frac{\sigma_\mu u}{\sigma_\nu u} = \xi_{\mu\nu}$, etc., the last equation beco

$$\frac{d\xi_{0\lambda}}{du} = \xi_{\mu\lambda}\,.\,\xi_{\nu\lambda},\quad \frac{d\xi_{\mu\nu}}{du} = -(e_\mu - e_\nu)\xi_{\lambda\nu}\,.\,\xi_{0\nu},\quad \frac{d\xi_{\lambda 0}}{du} = -\xi_{\mu 0}\xi_{\nu 0}.$$

For $u = 0$ these functions ξ satisfy the conditions

$$\xi_{0\lambda} = 0,\quad \xi_{\mu\nu} = 1,\quad \xi_{\lambda 0} = \infty.$$

From

$$\wp u - e_\lambda = \left(\frac{\sigma_\lambda u}{\sigma u}\right)^2,\ \lambda = 1,\ 2,\ 3,\ = \lambda,\ \mu,\ \nu,$$

we obtain

$$\sigma_\mu^2 u - \sigma_\nu^2 u + (e_\mu - e_\nu)\sigma^2 u = 0,$$

$$\sigma_\nu^2 u - \sigma_\lambda^2 u + (e_\nu - e_\lambda)\sigma^2 u = 0,$$

$$\sigma_\lambda^2 u - \sigma_\mu^2 u + (e_\lambda - e_\mu)\sigma^2 u = 0,$$

$$(e_\mu - e_\nu)\sigma_\lambda u + (e_\nu - e_\lambda)\sigma_\mu u + (e_\lambda - e_\mu)\sigma_\nu u = 0.$$

The differential equations are then thus transformed

$$\left(\frac{d\xi_{0\lambda}}{du}\right)^2 = \left(\frac{d}{du}\frac{\sigma}{\sigma_\lambda}\right)^2 = \xi_{\mu\lambda}^2\,.\,\xi_{\nu\lambda}^2 = \frac{\sigma_\mu^2\sigma_\nu^2}{\sigma_\lambda^2\sigma_\lambda^2}$$

$$= \frac{[\sigma_\lambda^2 + (e_\lambda - e_\mu)\sigma^2][\sigma_\lambda^2 - (e_\nu - e_\lambda)\sigma^2]}{\sigma_\lambda^2\,.\,\sigma_\lambda^2},$$

or
$$\left(\frac{d}{du}\xi_{0\lambda}\right)^2=\left[1-(e_\mu-e_\lambda)\left(\frac{\sigma}{\sigma_\lambda}\right)^2\right]\left[1-(e_\nu-e_\lambda)\left(\frac{\sigma}{\sigma_\lambda}\right)^2\right],$$
and similarly
$$\left(\frac{d}{du}.\xi_{\mu\nu}\right)^2=[1-\xi^2_{\mu\nu}][e_\mu-e_\lambda+(e_\lambda-e_\nu)\xi^2_{\mu\nu}],$$
$$\left(\frac{d}{du}.\xi_{\lambda 0}\right)^2=[\xi^2_{\lambda 0}-e_\lambda-e_\mu][\xi^2_{\lambda 0}+e_\lambda-e_\nu],$$
and, in general, the four functions
$$\frac{\sigma u}{\sigma_\lambda u},\quad \frac{1}{\sqrt{e_\mu-e_\lambda}}\cdot\frac{\sigma_\mu u}{\sigma_\nu u},\quad \frac{1}{\sqrt{e_\nu-e_\lambda}}\cdot\frac{\sigma_\nu u}{\sigma_\mu u},\quad \frac{1}{\sqrt{e_\mu-e_\nu}\sqrt{e_\nu-e_\lambda}}\cdot\frac{\sigma_\lambda u}{\sigma u},$$
satisfy the same differential equation
$$\left(\frac{d\xi}{du}\right)^2=(1-(e_\mu-e_\lambda)\xi^2)(1-(e_\nu-e_\lambda)\xi^2).$$
In order to compare these with the Jacobian differential equation, we have only to write
$$\sqrt{e_\lambda-e_\mu}\,\xi_{0\lambda}=\xi,\quad u_1=\sqrt{e_\lambda-e_\mu}\,.\,u,\quad \frac{e_\nu-e_\lambda}{e_\mu-e_\lambda}=k^2$$
whereupon
$$\frac{\xi}{\sqrt{e_\lambda-e_\mu}}=\xi_{0\lambda}=\frac{\sigma u}{\sigma_\lambda u}=\frac{\text{sn}\,u_1}{\sqrt{e_\lambda-e_\mu}}=\frac{\text{sn}\,(\sqrt{e_\lambda-e_\mu}\,.\,u,\,k)}{\sqrt{e_\lambda-e_\mu}},$$
and in a similar manner all the twelve sigma-quotients are produced,

$$\frac{\sigma u}{\sigma_3 u}=\frac{1}{\sqrt{e_1-e_3}}\text{sn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)$$
$$\frac{\sigma_1 u}{\sigma_3 u}=\text{cn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)$$
$$\frac{\sigma_2 u}{\sigma_3 u}=\text{dn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)$$

$$\frac{\sigma_1 u}{\sigma_2 u}=\text{sn coam}\,(\sqrt{e_1-e_3}\,.\,u,\,k)$$
$$\frac{\sigma_1 u}{\sigma_2 u}=\frac{1}{\sqrt{e_1-e_2}}\cos\text{coam}\,(\sqrt{e_1-e_3}\,.\,u,\,k)$$
$$\frac{\sigma_3 u}{\sigma_2 u}=\frac{\sqrt{e_1-e_3}}{\sqrt{e_1-e_2}}\Delta\,\text{coam}\,(\sqrt{e_1-e_3}\,.\,u,\,k)$$

$$\frac{\sigma_1 u}{\sigma u}=\sqrt{e_1-e_3}\frac{\text{cn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)}{\text{sn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)}$$
$$\frac{\sigma_2 u}{\sigma u}=\sqrt{e_1-e_3}\frac{\text{dn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)}{\text{sn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)}$$
$$\frac{\sigma_3 u}{\sigma u}=\sqrt{e_1-e_3}\frac{1}{\text{sn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)}$$

$$\frac{\sigma u}{\sigma_1 u}=\frac{1}{\sqrt{e-e}}\text{tn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)$$
$$\frac{\sigma_2 u}{\sigma_1 u}=\frac{1}{\sin\text{coam}\,(\sqrt{e_1-e_3}\,.\,u,\,k)}$$
$$\frac{\sigma_3 u}{\sigma_1 u}=\frac{1}{\text{cn}\,(\sqrt{e_1-e_3}\,.\,u,\,k)}$$

$$\text{coam}\,(\sqrt{e_1-e_3}\,.\,u,\,k)=\text{am}\,(K-\sqrt{e_1-e_3}\,.\,u,\,k).$$
Abel writes (Oeuvres, t. I, p. 265, nouvelle édition),
$$u=\int_0\frac{dx}{\sqrt{(1-c^2x^2)(1+e^2x^2)}},$$
$$x=\phi u,\quad \sqrt{1-c^2x^2}=fu,\quad \sqrt{1+e^2x^2}=Fu,$$

comparing which with the Weierstrassian notation,

$$x = \phi u = \frac{\sigma u}{\sigma_2 u}, \quad fu = \frac{\sigma_1 u}{\sigma_2 u}, \quad Fu = \frac{\sigma_3 u}{\sigma_2 u},$$

if only

$$e_1 - e_2 = -c^2, \quad -(e_3 - e_2) = e^2.$$

As regards the analogues of Jacobi's K and K, it is to be noticed that, as usually defined by the equations

$$K = \int_0^1 \frac{dt}{\sqrt{1-t^2} . \sqrt{1-k^2t^2}}, \quad K' = \int_0^1 \frac{dt}{\sqrt{1-t^2} . \sqrt{1-k'^2t^2}},$$

the values are only unambiguous when the path of integration is fixed, it being generally understood that the path of integration is the straight line from 0 to 1. Corresponding to this we have, *e. g.*,

$$K = \sqrt{e_1 - e_3}(\omega + 4p\omega + 2q\omega'),$$

where the determination of the path of integration corresponds to the freedom of choice of p and q. Commonly we have $p = q = 0$, and

$$K = \omega\sqrt{e_1 - e_3}, \quad K'i = \omega'\sqrt{e_1 - e_3},$$

and then 2ω, $2\omega'$ form a primitive period-pair for the function $\wp(u, g_1, g_2)$, and, if we write as before $\omega + \omega' = \omega''$, or $\omega_1 + \omega_3 = \omega_2$, then is $\wp\omega_1 = e_1$, $\wp\omega_2 = e_2$, $\wp\omega_3 = e_3$.

The functions $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$, can be represented as an infinite product of the same form as that for σu by writing

$$w_1 = (2\mu + 1)\omega + 2\mu'\omega', \quad w_2 = (2\mu + 1)\omega + (2\mu' + 1)\omega', \quad w_3 = 2\mu\omega + (2\mu + 1)\omega',$$

where μ, $\mu' = 0, \pm 1, \pm 2, \ldots \pm \infty$; namely,

$$\sigma_\lambda u = e^{-\frac{1}{2}e_\lambda u^2} \Pi_{w_\lambda}\left(1 - \frac{u}{w_\lambda}\right) e^{\frac{u}{w_\lambda} + \frac{1}{2}\frac{u^2}{w_\lambda^2}}.$$

But these functions are also representible in the form of singly infinite products. As an aid in transforming, Professor Schwarz makes use of the following table. When the argument u assumes the values $u + \omega$, $u + \omega'$, $u + \omega''$, then the magnitudes $v = \frac{u}{2\omega}$, $z = e^{v\pi i}$, $2\eta\omega v^2$, $e^{2\eta\omega v^2}$, assume the values in the table, where $\tau = \frac{\omega'}{\omega}$, $h = e^{\tau\pi i}$,

u	$u + \omega$	$u + \omega'$	$u + \omega''$
v	$v + \frac{1}{2}$	$v + \frac{1}{2}\tau$	$v + \frac{1}{2} + \frac{1}{2}\tau$
z	iz	$h^{\frac{1}{2}} . z$	$i . h^{\frac{1}{2}} . z$
$2\eta\omega v^2$	$2\eta\omega v^2 + \eta u + \frac{1}{2}\eta\omega$	$2\eta\omega v^2 + \eta' u + \frac{1}{2}\eta'\omega' + \frac{1}{4}\tau\pi i + v\pi i$	$2\eta\omega v^2 + \eta'' u + \frac{1}{2}\eta''\omega'' + \frac{1}{4}\pi i + \frac{1}{4}\tau\pi i + v\pi i$
$e^{2\eta\omega v^2}$	$e^{2\eta\omega v^2} . e^{\eta u} . e^{\frac{1}{2}\eta\omega}$	$e^{2\eta\omega v^2} . e^{\eta' u} . e^{\frac{1}{2}\eta'\omega'} . h^{\frac{1}{4}} . z$	$e^{2\eta\omega v^2} . e^{\eta'' u} . e^{\frac{1}{2}\eta''\omega''} . \sqrt{i} . h^{\frac{1}{4}} . z$

With the help of this table the infinite product for σu on page 261, Vol. VI of this Journal, can be transformed as follows: Developing

$$(1-h^{2n}.z^2)(1-h^{2n}z^{-2}) = 1-2h^{2n}\cos\frac{u\pi}{\omega}+h^{4n}$$

since

$$1-\cos\frac{u\pi}{\omega} = 2\sin^2\frac{u\pi}{2\omega}$$

$$\begin{aligned}(1-h^{2n}z^2)(1-h^{2n}z^{-2}) &= 1-2h^{2n}+4h^{2n}.\sin^2\frac{u\pi}{\omega}+h^{4n}\\ &= (1-h^{2n})^2+4h^{2n}.\sin^2\frac{u\pi}{2\omega}\\ &= (1-h^{2n})^2\left\{1+\left(\frac{2}{h^n-h^{-n}}\right)^2.\sin^2\frac{u\pi}{2\omega}\right\},\end{aligned}$$

but $\dfrac{h^n-h^{-n}}{2} = i\sin n\dfrac{\pi\omega'}{\omega}$, consequently

$$(1-h^{2n}z^2)(1-h^{2n}z^{-2}) = (1-h^{2n})^2\left\{1-\frac{\sin^2\frac{u\pi}{2\omega}}{\sin^2 n\frac{\pi\omega'}{\omega}}\right\}$$

and

$$\sigma u = \frac{2\omega}{\pi}\cdot e^{\frac{\eta u^2}{2\omega}}\cdot\frac{z-z^{-1}}{2i}\Pi_n\frac{1-h^{2n}z^2}{1-h^{2n}}\cdot\frac{1-h^{2n}z^{-2}}{1-h^{2n}},$$

which is the desired expression for σu. Since further

$$\sigma_1 u = e^{-\eta u}\cdot\frac{\sigma(u+\omega)}{\sigma\omega},$$

we obtain by the assistance of the table the analogous expressions for σ_1, σ_2, σ_3; namely

$$\begin{aligned}\sigma_1 u &= e^{2\eta\omega v^2}.\cos v\pi\,\Pi_n\frac{\cos(n\pi-v)\pi}{\cos n\tau\pi}.e^{-v\pi i}.\Pi_n\frac{\cos(n\tau+v)\pi}{\cos n\tau\pi}.e^{v\pi i}\\ &= e^{2\eta\omega v^2}.\frac{z+z^{-1}}{2}.\Pi_n\frac{1+h^{2n}z^{-2}}{1+h^{2n}}.\Pi_n\frac{1+h^{2n}z^2}{1+h^{2n}}\\ &= e^{2\eta\omega v^2}.\cos v\pi.\Pi_n\frac{1+2h^{2n}\cos 2v\pi+h^{4n}}{(1+h^{2n})^2},\end{aligned}$$

$$\begin{aligned}\sigma_2 u &= e^{2\eta\omega v^2}.\Pi_n\frac{\cos((n-\frac12)\tau-v)\pi}{\cos(n-\frac12)\tau\pi}.e^{-v\pi i}.\Pi_n\frac{\cos((n-\frac12)\tau+v)\pi}{\cos(n-\frac12)\tau\pi}.e^{v\pi i}\\ &= e^{2\eta\omega v^2}.\Pi_n\frac{1+h^{2n-1}.z^{-2}}{1-h^{2n-1}}.\Pi_n\frac{1+h^{2n-1}.z^2}{1+h^{2n-1}}\\ &= e^{2\eta\omega v^2}.\Pi_n\frac{1+2h^{2n-1}.\cos 2v\pi+h^{4n-2}}{(1+h^{2n-1})^2},\end{aligned}$$

$$\begin{aligned}\sigma_3 u &= e^{2\eta\omega v^2}.\Pi_n\frac{\sin((n-\frac12)\tau-v)\pi}{\sin(n-\frac12)\tau\pi}.e^{-v\pi i}.\Pi_n\frac{\sin((n-\frac12)\tau+v)\pi}{\sin(n-\frac12)\tau\pi}.e^{v\pi i}\\ &= e^{2\eta\omega v^2}.\Pi_n\frac{1-h^{2n-1}z^{-2}}{1-h^{2n-1}}.\Pi_n\frac{1-h^{2n-1}z^2}{1-h^{2n-1}}\\ &= e^{2\eta\omega v^2}.\Pi_n\frac{1-2h^{2n-1}.\cos 2v\pi+h^{4n-2}}{(1-h^{2n-1})^2}.\end{aligned}$$

Analogous to the expression for σu at the bottom of p. 261, we have

$$\sigma_1 u = e^{\frac{\eta u^2}{2\omega}}.\cos\frac{u\pi}{2\omega}.\Pi_n\left(1-\frac{\sin^2\frac{u\pi}{2\omega}}{\cos^2 n\frac{\omega'\pi}{\omega}}\right)$$

$$\sigma_2 u = e^{\frac{\eta u^2}{2\omega}}.\Pi_n\left(1-\frac{\sin^2\frac{u\pi}{2\omega}}{\cos^2(n-\frac{1}{2})\frac{\omega'\pi}{\omega}}\right)$$

$$\sigma_3 u = e^{\frac{\eta u^2}{2\omega}}.\Pi_n\left(1-\frac{\sin^2\frac{u\pi}{2\omega}}{\sin^2(n-\frac{1}{2})\frac{\omega'\pi}{\omega}}\right).$$

In the normal case we shall have ω real and $\frac{\omega'}{\omega}$ imaginary, and therefore none of the quotients under the product sign can assume the value unity. The functions σu and $\sigma_1 u$ disappear accordingly only when $\sin\frac{u\pi}{2\omega}$ and $\cos\frac{u\pi}{2\omega}$ respectively vanish. The Jacobian functions $\frac{\sigma}{\sigma_3}$, $\frac{\sigma_1}{\sigma_3}$, $\frac{\sigma_2}{\sigma_3}$, are analogous, the first to the sine, the second to the cosine, while the third remains positive for real values of u. The Abelian forms $\frac{\sigma}{\sigma_2}$, $\frac{\sigma_1}{\sigma_2}$, $\frac{\sigma_3}{\sigma_2}$, are analogous, the first to the tangent, the second and third to the secant.

The expressions for the root-differences and the connection with the ϑ-functions are obtained in the following manner. Defining as above $h = e^{\frac{\omega'}{\omega}\pi i} = e^{\tau\pi i}$, and writing

$$h_0 = \prod_{n=1}^{\infty}(1-h^{2n}),\ h_1 = \prod_{n=1}^{\infty}(1+h^{2n}),\ h_2 = \prod_{n=1}^{\infty}(1+h^{2n-1}),\ h_3 = \prod_{n=1}^{\infty}(1-h^{2n-1}),$$

then is

$$h_0 = h_0.h_1.h_2.h_3;\quad h_1.h_2.h_3 = 1.$$

For

$$\begin{aligned} h_0 &= (1-h^2)(1-h^4)(1-h^6)\ldots \\ &= (1+h)(1+h^2)(1+h^3)\ldots \\ &\quad .(1-h)(1-h^2)(1-h^3)\ldots \end{aligned}$$

and the proof is apparent. With the aid of these facts the relation between the periods and the root-differences is easily discovered. Starting again from the "pocket-edition"

$$\wp u - \wp v = \frac{\sigma(u+v)\,\sigma(v-u)}{\sigma^2 u\ \sigma^2 v},$$

and writing $u=\omega$, $v=\omega'$, $\omega''=\omega+\omega'$, the equation becomes

$$e_1 - e_3 = \frac{\sigma\omega''.\,\sigma(\omega'-\omega)}{\sigma^2\omega\ \sigma^2\omega'}.$$

But noticing that $\omega' - \omega = \omega'' - 2\omega$, $\omega'' - \omega = \omega'$, and that

$$\sigma(\omega'' - 2\omega) = -\sigma(-\omega'' + 2\omega) = -\sigma\omega''.e^{-2\eta(\omega'-\omega)},$$

$$\sigma(\omega' - \omega) = \sigma(\omega'' - 2\omega) = -\sigma\omega''.e^{-2\eta(\omega''-\omega)},$$

the equation becomes

$$e_1 - e_3 = -\left(\frac{\sigma\omega''}{\sigma\omega\,\sigma\omega'}\right)^2.e^{-2\eta\omega'}.$$

And in a similar manner, writing $u = \omega$, $v = \omega + \omega'$, we have,

$$e_1 - e_2 = \frac{\sigma(\omega' + 2\omega)\,\sigma(\omega')}{\sigma^2\omega\,\sigma^2\omega''} = -\left(\frac{\sigma\omega'}{\sigma\omega\,\sigma\omega''}\right)^2.e^{2\eta\omega''},$$

and for $u = \omega''$, $v = \omega'$,

$$e_2 - e_3 = \frac{\sigma(\omega + 2\omega'')\,\sigma(\omega)}{\sigma^2\omega''\,\sigma^2\omega'} = \left(\frac{\sigma\omega'}{\sigma\omega\,\sigma\omega''}\right)^2.e^{2\eta'\omega''}.$$

But these formulæ can be still further simplified. From

$$\sigma u = \frac{2\omega}{\pi}\,e^{\frac{\eta u^2}{2\omega}}.\mathrm{sn}\,\frac{u\pi'}{2\omega}.\Pi\,\frac{1 - h^{2n}z^2}{1 - h^{2n}}\cdot\frac{1 - h^{2n}z^{-2}}{1 - h^{2n}},$$

we have for $u = \omega$

$$\sigma\omega = \frac{2\omega}{\pi}\,e^{\frac{\eta\omega}{2}}.\Pi\,\frac{1 + h^{2n}}{1 - h^{2n}}\cdot\frac{1 + h^{2n}}{1 - h^{2n}} = \frac{2\omega}{\pi}\,e^{\frac{\eta\omega}{2}}.\frac{h_1^2}{h_0^2}.$$

And similarly

$$\sigma\omega' = e^{\eta\frac{\omega'^2}{2\omega}}.\frac{1}{1 - h}.h^{-\frac{1}{4}}.\frac{h^{-1}}{2i}\cdot\frac{h_3^2}{h_0},$$

$$\sigma\omega'' = \frac{2\omega}{\pi}\,e^{\frac{(\eta+\eta')\omega''}{2}}\sqrt{\frac{i}{2}}.h^{-\frac{1}{4}}.\frac{h_2^2}{h_0^2}.$$

The expressions for the root-differences become then

$$\left(\frac{2\omega}{\pi}\right)^2(e_1 - e_3) = \left(\frac{h_2h_0}{h_1h_3}\right)^4 = h_0^4h_2^8,$$

$$\left(\frac{2\omega}{\pi}\right)^2(e_1 - e_2) = \left(\frac{h_3^2h_0^2}{h_2^2h_1^2}\right)^2 = h_0^4h_3^8,$$

$$\left(\frac{2\omega}{\pi^2}\right)^2(e_2 - e_3) = \qquad 16.h.h_0^4h_1^8,$$

where $h = e^{\frac{\omega'}{\omega}\pi i}$, and h_0, h_1, h_2, h_3, are defined above. In accordance also with previous definitions for the k and k' of Jacobi,

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = 16h\left\{\frac{(1 + h^2)(1 + h^4)(1 + h^6)\ldots}{(1 + h)(1 + h^3)(1 + h^5)\ldots}\right\}^8$$

$$k'^2 = \frac{e_1 - e_2}{e_1 - e_3} = \left\{\frac{(1 - h)(1 - h^3)(1 - h^5)\ldots}{(1 + h)(1 + h^3)(1 + h^5)\ldots}\right\}^8.$$

The four sigma-functions are now expressed through the functions ϑ and Θ, as follows. The infinite product $F(z) = \Pi_n(1 - h^{2n})(1 + h^{2n-1}z^{-2})(1 + h^{2n-1}z^{+2})$ can be expressed as a power series of z^2 which converges for all values of z except

$z=0$, so long as $h<1$. This is plain from the following identity,

$$\Pi_n(1-h^{2n})(1+h^{2n-1}z^{-2})(1+h^{2n-1}z^2)$$
$$=1+h(z^2+z^{-2})+h^4(z^4+z^{-4})+h^9(z^6+z^{-6})+\ldots$$

The development of the sigma-function follows at once, since

$$h_2^2.h_0.\sigma_2 u=e^{\frac{\eta u^2}{2\omega}}.F(z)\quad\text{and}\sqrt{\frac{2\omega}{\pi}}\sqrt{e_1-e_3}=h_0.h_2^2.$$

We have then

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1-e_3}.\sigma_2 u=e^{\frac{\eta u^2}{2\omega}}F(z)=e^{\frac{\eta u^2}{2\omega}}.\sum_n h^{n^2}z^{2n}$$

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1-e_2}.\sigma_3 u=e^{\frac{\eta u^2}{2\omega}}F(zi)=e^{\frac{\eta u^2}{2\omega}}.\sum_n(-1)^n h^{n^2}z^{2n}$$

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_2-e_3}.\sigma_1 u=e^{\frac{\eta u^2}{2\omega}}.h^{\frac{1}{4}}.z.F(zh^{\frac{1}{2}})=e^{\frac{\eta u^2}{2\omega}}.\sum_n h^{\frac{1}{4}(2n+1)^2}z^{2n+1}$$

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1-e_3}\sqrt[4]{e_1-e_2}.\sqrt[4]{e_2-e_3}.\sigma u=\frac{1}{i}e^{\frac{\eta u^2}{2\omega}}h^{\frac{1}{4}}z.F(izh^{\frac{1}{2}}).$$

It will be remembered that e_1, e_2, e_3 are the roots of the equation

$$4x^3-g2x-g_3=0.$$

The discriminant of which, squared

$$\{(e_1-e_2)(e_2-e_3)(e_3-e_2)\}^2=\frac{g_2^3-27g_3^2}{16}=G,$$

so that $$\sqrt{\frac{2\omega}{\pi}}\cdot\sqrt[8]{G}=\frac{1}{i}e^{\frac{\eta u^2}{2\omega}}h^{\frac{1}{4}}.z.F(h^{\frac{1}{2}}z)=e^{\frac{\eta u^2}{2\omega}}\frac{1}{i}\sum_n(-1)^n h^{\frac{1}{4}(2n+1)^2}z^{2n+1}$$

$$n=0,\quad\pm1,\quad\pm2,\ldots\pm\infty,$$

$$z=e^{\frac{u\pi i}{2\omega}},\quad h=e^{\frac{\omega'}{\omega}\pi i}.$$

The expression for Fz becomes, since

$$z^{2n}+z^{-2n}=2\cos\frac{nu\pi i}{\omega}.$$

$$F(z)=1+2h\cos\frac{u\pi i}{\omega}+2h^4\cos 2\frac{u\pi i}{\omega}+2h^9\cos 3\frac{u\pi i}{\omega}+\ldots$$

which is the ϑ series of Jacobi (*Werke*, Bd. I, p. 501). Weierstrass defines the ϑ functions as follows:

$$\frac{1}{i}\sum_n(-1)^n h^{\frac{1}{4}(2n+1)^2}z^{2n+1}=2h^{\frac{1}{4}}\sin v\pi-2h^{\frac{9}{4}}\sin 3v\pi+2h^{\frac{25}{4}}\sin 5v\pi=\ldots=\vartheta_1(v),$$

$$\sum_n h^{\frac{1}{4}(2n+1)^2}z^{2n+1}=2h^{\frac{1}{4}}\cos v\pi+2h^{\frac{9}{4}}\cos 3v\pi+2h^{\frac{25}{4}}\cos 5v\pi+\ldots=\vartheta_2(v),$$

$$\sum_n h^{n^2}z^{2n}=1+2h\cos 2v\pi+2h^4\cos 4v\pi+2h^9\cos 6v\pi+\ldots=\vartheta_3(v),$$

$$\sum_n(-1)^n h^{n^2}z^{2n}=1-2h\cos 2v\pi+2h^4\cos 4v\pi-2h^9\cos 6v\pi+\ldots\doteq\vartheta_0(v),$$

which agree with Jacobi's notation when $v\pi = x$; $h = q$. Hermite writes

$$\sum_m (-1)^{m\nu} e^{i\pi[(2m+\mu)x + \frac{1}{4}\omega(2m+\mu)^2]} = \theta_{\mu,\nu}(x),$$

$$m = 0, \quad \pm 1, \quad \pm 2, \ldots \pm \infty.$$

If now we define the function Θ by the equation

$$\Theta_\nu(u) = e^{2\eta\omega v^2} . \vartheta_\nu\left(v, \frac{\omega'}{\omega}\right); \; u = 2\omega v,$$

then the four sigma-functions are thus expressed through Θ,

$$\sqrt{\frac{2\omega}{\pi}} \sqrt[8]{G} . \sigma u = e^{2\eta\omega v^2} \vartheta_1\left(v, \frac{\omega'}{\omega}\right) = \Theta_1(u, \omega, \omega'),$$

$$\sqrt{\frac{2\omega}{\pi}} \sqrt[4]{e_\mu - e_\nu} . \sigma_\lambda u = e^{2\eta\omega v^2} \vartheta_2\left(v, \frac{\omega'}{\omega}\right) = \Theta_2(u, \omega, \omega'),$$

$$\sqrt{\frac{2\omega}{\pi}} \sqrt[4]{e_\lambda - e_\nu} . \sigma_\mu u = e^{2\eta\omega v^2} \vartheta_3\left(v, \frac{\omega'}{\omega}\right) = \Theta_3(u, \omega, \omega'),$$

$$\sqrt{\frac{2\omega}{\pi}} \sqrt[4]{e_\lambda - e_\mu} . \sigma_\nu u = e^{2\eta\omega v^2} \vartheta_0\left(v, \frac{\omega'}{\omega}\right) = \Theta_0(u, \omega, \omega'),$$

where again $v = \frac{u}{2\omega}$. By developing according to powers of v, and comparing we have

$$\sqrt{\frac{2\omega}{\pi}} \sqrt[8]{G} = \frac{1}{2\omega} \vartheta'_1(0) = \frac{\pi}{\omega} h^{\frac{1}{4}}(1 - 3h^{1.2} + 5h^{2.3} - 7h^{3.4} + \ldots)$$

$$\sqrt{\frac{2\omega}{\pi}} \sqrt[4]{e_\mu - e_\nu} = \vartheta_2(0) = 2h^{\frac{1}{4}}(1 + h^{1.2} + h^{2.3} + h^{3.4} + \ldots)$$

$$\sqrt{\frac{2\omega}{\pi}} \sqrt[4]{e_\lambda - e_\nu} = \vartheta_3(0) = 1 + 2h + 2h^4 + 2h^9 + \ldots$$

$$\sqrt{\frac{2\omega}{\pi}} \sqrt[4]{e_\lambda - e_\mu} = \vartheta_0(0) = 1 - 2h + 2h^4 - 2h^9 + \ldots$$

From these spring the following equations which become useful in computation:

$$\vartheta'_1(0) = \pi\vartheta_0(0) . \vartheta_2(0) . \vartheta_3(0), \quad \vartheta_0^4(0) + \vartheta_2^4(0) = \vartheta_3^4(0),$$

$$e_\lambda = \frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 (\vartheta_3^4(0) + \vartheta_0^4(0)), \quad e_\mu = \frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 (\vartheta_2^4(0) - \vartheta_0^4(0)),$$

$$e_\nu = -\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 (\vartheta_2^4(0) + \vartheta_3^4(0)),$$

$$\sqrt{\frac{2\omega}{\pi}} = \frac{2h^{\frac{1}{4}} + 2h^{\frac{9}{4}} + 2h^{\frac{25}{4}} + \ldots}{\sqrt[4]{e_\mu - e_\nu}} = \frac{2}{\sqrt[4]{e_\lambda - e_\nu} - \sqrt[4]{e_\lambda - e_\mu}} (2h + 2h^9 + 2h^{25} + \ldots)$$

$$\sqrt{\frac{2\omega}{\pi}} = \frac{1 + 2h + 2h^4 + 2h^9 + \ldots}{\sqrt[4]{e_\lambda - e_\nu}} = \frac{2}{\sqrt[4]{e_\lambda - e_\nu} + \sqrt[4]{e_\lambda - e_\mu}} (1 + 2h^4 + 2h^{16} + \ldots)$$

$$\sqrt{k} = \frac{\sqrt[4]{e_\mu - e_\nu}}{\sqrt[4]{e_\lambda - e_\nu}} = \frac{\vartheta_2\left(0, \frac{\omega'}{\omega}\right)}{\vartheta_3\left(0, \frac{\omega'}{\omega}\right)} = \frac{2h^{\frac{1}{4}} + 2h^{\frac{9}{4}} + 2h^{\frac{25}{4}} + \ldots}{1 + 2h + 2h^4 + 2h^9 + \ldots}$$

$$\sqrt{k'} = \frac{\sqrt[4]{e_\lambda - e_\mu}}{\sqrt[4]{e_\lambda - e_\nu}} = \frac{\vartheta_0\left(0, \frac{\omega'}{\omega}\right)}{\vartheta_3\left(0, \frac{\omega'}{\omega}\right)} = \frac{1 - 2h + 2h^4 - 2h^9 + \dots}{1 + 2h + 2h^4 + 2h^9 + \dots}.$$

We define

$$l = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} = \frac{\sqrt[4]{e_\lambda - e_\nu} - \sqrt[4]{e_\lambda - e_\mu}}{\sqrt[4]{e_\lambda - e_\nu} + \sqrt[4]{e_\lambda - e_\mu}} = \frac{2h + 2h^9 + \dots}{1 + 2h^4 + 2h^{16} + \dots} = \frac{\vartheta_2\left(0, 4\frac{\omega'}{\omega}\right)}{\vartheta_3\left(0, 4\frac{\omega'}{\omega}\right)}$$

which is identically satisfied by writing

$$h = q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + 150\left(\frac{l}{2}\right)^{13} + \dots$$

These expressions for l and h make the computation of the period 2ω or of Jacobi's K very easy.

In explaining more at length the methods employed for computing, I cannot do better than to give them with scarcely any variation from the words of that most genial expounder of Weierstrass' theories, Prof. Schwarz. When the three roots e_1, e_2, e_3, are once known, we can, by the aid of the formulæ just given, not only compute with the greatest ease the two periods 2ω, $2\omega'$, but we can also express the sigma-quotients through such ϑ series that the argument h shall have the smallest possible value, and the series converge most rapidly. This last end is brought about by so choosing the order of magnitude of e_1, e_2, e_3, that

$$l = \frac{\sqrt[4]{e_\lambda - e_\nu} - \sqrt[4]{e_\lambda - e_\mu}}{\sqrt[4]{e_\lambda - e_\nu} + \sqrt[4]{e_\lambda - e_\mu}}$$

which is used in the computation of h shall be as small as possible. Of the several cases which present themselves according as the invariants g_2, g_3 and the roots e_1, e_2, e_3, are real or imaginary, I shall discuss here but one, where all are real. The roots will be real when the discriminant $G = \frac{1}{16}(g_2^3 - 27g_3^2)$ of the cubic equation $4s^3 - g_2 s - g_3 = 0$ is positive and g_2 and g_3 real. We then assume $e_1 > e_2 > e_3$ and all the radicals positive; farther $\lambda = 1$, $\mu = 2$, $\nu = 3$, whereupon

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad k'^2 = \frac{e_1 - e_2}{e_1 - e_3}, \quad \omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega_3 = \frac{K'i}{\sqrt{e_1 - e_3}},$$

$$\frac{\sigma'\omega_1}{\sigma\omega_1} = \eta_1, \quad \frac{\sigma'\omega_3}{\sigma\omega_3} = \eta_3, \quad \tau = \frac{\omega_3}{\omega_1}, \quad h = e^{\tau\pi i}, \quad h_1 = h' = e^{-\frac{\pi i}{\tau}},$$

$$v = \frac{u}{2\omega_1}, \quad v_1 = \frac{ui}{2\omega_3}, \quad \text{where } \omega_1, \ \frac{\omega_3}{i}, \ \frac{\tau}{i}, \ h_1 h, \text{ are positive.}$$

For the computation of the periods the following system of equations is used:

$$l = \frac{\sqrt[4]{e_1 - e_3} - \sqrt[4]{e_1 - e_2}}{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}}, \quad h = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + 150\left(\frac{l}{2}\right)^{13} + \dots$$

$$\sqrt{\frac{2\omega_1}{\pi}} = \frac{2}{\sqrt[4]{e_1-e_3}+\sqrt[4]{e_1-e_2}}(1+2h^4+2h^{16}+\ldots), \quad \omega_3 = \frac{\omega_1 i}{\pi}\,\text{log. nat.}\,\frac{1}{h},$$

$$2\eta_1\omega_1 = \frac{\pi^2}{6}\,\frac{1-3^3h^2+5^3h^6-7^3h^{12}+\ldots}{1-3h^2+5h^6-7h^{12}+\ldots}, \quad \eta_1\omega_3-\omega_1\eta_3 = \tfrac{1}{2}\pi i,$$

$$\sqrt{\frac{2\omega_1}{\pi}}\,\sqrt[4]{e_2-e_3} = 2h^{\frac{1}{4}}(1+h^2+h^6+h^{12}+\ldots)$$

$$\sqrt{\frac{2\omega_1}{\pi}}\,\sqrt[4]{e_1-e_3} = 1+2h+2h^4+2h^9+\ldots$$

$$\sqrt{\frac{2\omega_1}{\pi}}\,\sqrt[4]{e_1-e_2} = 1-2h+2h^4-2h^9+\ldots$$

$$\sqrt{\frac{2\omega_1}{\pi}}\,\sqrt[8]{G} = \frac{\pi}{\omega_1}\cdot h^{\frac{1}{4}}(1-3h^2+5h^6-7h^9+\ldots)$$

For the calculation of the sigma-functions we shall have

$$\sqrt{\frac{2\omega_1}{\pi}}\,\sqrt[8]{G}\,.\,\sigma u = e^{2\eta_1\omega_1 v^2}\,\vartheta_1(v,\tau)$$

$$\sqrt{\frac{2\omega_1}{\pi}}\,\sqrt[4]{e_2-e_3}\,.\,\sigma_1 u = e^{2\eta_1\omega_1 v^2}\,\vartheta_2(v,\tau)$$

$$\sqrt{\frac{2\omega_1}{\pi}}\,\sqrt[4]{e_1-e_3}\,.\,\sigma_2 u = e^{2\eta_1\omega_1 v^2}\,\vartheta_3(v,\tau)$$

$$\sqrt{\frac{2\omega_1}{\pi}}\,\sqrt[4]{e_1-e_2}\,.\,\sigma_3 u = e^{2\eta_1\omega_1 v^2}\,\vartheta_0(v,\tau)$$

$$\vartheta_0(v,\tau) = 1-2h\cos 2v\pi+2h^4\cos 4v\pi-2h^9\cos 6v\pi+\ldots$$
$$\vartheta_1(v,\tau) = 2h^{\frac{1}{4}}\sin v\pi-2h^{\frac{9}{4}}\sin 3v\pi+2h^{\frac{25}{4}}\sin 5v\pi-\ldots$$
$$\vartheta_2(v,\tau) = 2h^{\frac{1}{4}}\cos v\pi+2h^{\frac{9}{4}}\cos 3v\pi+2h^{\frac{25}{4}}\cos 5v\pi+\ldots$$
$$\vartheta_3(v,\tau) = 1+2h\cos 2v\pi+2h^4\cos 4v\pi+2h^9\cos 6v\pi+\ldots$$

If, however, we have to choose $\lambda=3$, $\mu=2$, $\nu=1$, the equations become

$$l_1 = \frac{\sqrt[4]{e_1-e_3}-\sqrt[4]{e_2-e_3}}{\sqrt[4]{e_1-e_3}+\sqrt[4]{e_2-e_3}}, \quad h_1 = \frac{l_1}{2}+2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l}{2}\right)^9+150\left(\frac{l}{2}\right)^{13}+\ldots$$

$$\sqrt{\frac{2\omega_3}{\pi i}} = \frac{2}{\sqrt[4]{e_1-e_3}+\sqrt[4]{e_2-e_3}}(1+2h_1^4+2h_1^{16}+\ldots)\quad \omega_1 = \frac{\omega_3}{\pi i}\,\text{log. nat.}\left(\frac{1}{h_1}\right)$$

$$\sqrt{\frac{2\omega_3}{\pi i}}\,.\,\sqrt[4]{e_1-e_2} = 2h_1^{\frac{1}{4}}(1+h_1^2+h_1^6+h_1^{12}+\ldots)$$

$$\sqrt{\frac{2\omega_3}{\pi i}}\,.\,\sqrt[4]{e_1-e_3} = 1+2h_1+2h_1^4+2h_1^9+\ldots$$

$$\sqrt{\frac{2\omega_3}{\pi i}}\,.\,\sqrt[4]{e_2-e_3} = 1-2h_1+2h_1^4-2h_1^9+\ldots$$

$$\sqrt{\frac{2\omega_3}{\pi i}}\,.\,\sqrt[8]{G} = \frac{\pi i}{\omega_3}\,h_1^{\frac{1}{4}}(1-3h_1^9+5h_1^6-7h_1^{12}+\ldots)$$

$$\sqrt{\frac{2\omega_3}{\pi i}}\sqrt[8]{G}.\sigma u=\frac{1}{i}e^{-2\eta_3\omega_3 v_1^2}\vartheta_1\left(v_1 i,\frac{-1}{\tau}\right),$$

$$\sqrt{\frac{2\omega_3}{\pi i}}\sqrt[4]{e_2-e_3}.\sigma_1 u=e^{-2\eta_3\omega_3 v_1^2}\vartheta_0\left(v_1 i,\frac{-1}{\tau}\right),$$

$$\sqrt{\frac{2\omega_3}{\pi i}}\sqrt[4]{e_1-e_3}.\sigma_2 u=e^{-2\eta_3\omega_3 v_1^2}\vartheta_3\left(v_1 i,\frac{-1}{\tau}\right),$$

$$\sqrt{\frac{2\omega_3}{\pi i}}\sqrt[4]{e_1-e_2}.\sigma_3 u=e^{-2\eta_3\omega_3 v_1^2}\vartheta_2\left(v_1 i,\frac{-1}{\tau}\right).$$

$$\vartheta_0\left(v_1 i,\frac{-1}{\tau}\right)=1-h_1(e^{2v_1\pi}+e^{-2v_1\pi})+h_1^4(e^{4v_1\pi}+e^{-4v_1\pi})-h_1^9 e^{6v_1\pi}+e^{-6v_1\pi})+\dots$$

$$\frac{1}{i}\vartheta_1\left(v_1 i,\frac{-1}{\tau}\right)=h_1^{\frac{1}{4}}(e^{v_1\pi}-e^{-v_1\pi})-h_1^{\frac{9}{4}}(e^{3v_1\pi}-e^{3v_1\pi})+h_1^{\frac{25}{4}}(e^{5v_1\pi}-e^{-5v_1\pi})-\dots$$

$$\vartheta_2\left(v_1 i,\frac{-1}{\tau}\right)=h_1^{\frac{1}{4}}(e^{v_1\pi}+e^{-v_1\pi})+h_1^{\frac{9}{4}}(e^{3v_1\pi}+e^{3v_1\pi})+h_1^{\frac{25}{4}}(e^{5v_1\pi}+e^{-5v_1\pi})+\dots$$

$$\vartheta_3\left(v_1 i,\frac{-1}{\tau}\right)=1+h_1(e^{2v_1\pi}+e^{-2v_1\pi})+h_1^4(e^{4v_1\pi}+e^{-4v_1\pi})+h_1^9(e^{6v_1\pi}+e^{-6v_1\pi})+\dots$$

When $e_2-e_3\lesseqgtr e_1-e_2$, that is, $e_2\lesseqgtr 0$, then is

$$l_1\gtreqless\frac{\sqrt[4]{2}-1}{\sqrt[4]{2}+1},\quad h_1\gtreqless e^{-\pi}\quad\text{and } l\lesseqgtr\frac{\sqrt[4]{2}-1}{\sqrt[4]{2}+1},\quad h\lesseqgtr e^{-\pi}$$

When g_3 is positive, that is $e_2<0$, it is advisable to use the formulæ for h and l, otherwise those for h_1, l_1 will be found best, because in the first case we have $h<h_1$, in the second $h_1<h$.

For the calculation of the elliptic integral of the first kind in the ordinary form, Professor Schwarz throws the necessary formulæ into the following shape "Among the values of u for which $\wp u=s$, there are, in consequence of the equation

$$\frac{\sigma_3(u\pm 2\omega')}{\sigma_2(u\pm 2\omega')}=-\frac{\sigma_3 u}{\sigma_2 u},$$

always such for which the real component of $\dfrac{\sqrt[4]{e_1-e_2}}{\sqrt[4]{e_1-e_3}}$ is not negative and consequently the modulus of

$$\frac{\sqrt[4]{e_1-e_3}.\sigma_2 u-\sqrt[4]{e_1-e_2}.\sigma_3 u}{\sqrt[4]{e_1-e_3}.\sigma_2 u+\sqrt[4]{e_1-e_3}.\sigma_3 u}=\frac{\sqrt[4]{e_1-e_3}-\sqrt[4]{e_1-e_2}}{\sqrt[4]{e_1-e_3}+\sqrt[4]{e_1-e_2}}\cdot\frac{\sigma_1(2u,\omega,4\omega')}{\sigma_2(2u,\omega,4\omega')},$$

is not greater than unity. The value of $\sqrt{s-e_2}$ can be chosen at pleasure, after which the value of $\sqrt{s-e_3}$ can be so taken that the real component of

$$\frac{\sqrt[4]{e_1-e_2}.\sqrt{s-e_2}}{\sqrt[4]{e_1-e_3}.\sqrt{s-e_2}}$$

shall not be negative. We then write

$$\frac{\sqrt[4]{e_1-e_3}-\sqrt[4]{e_1-e_2}}{\sqrt[4]{e_1-e_3}+\sqrt[4]{e_1-e_2}}=l, \quad \frac{\sqrt[4]{e_1-e_3}\,.\sqrt{s-e_2}-\sqrt[4]{e_1-e_2}\,.\sqrt{s-e_3}}{\sqrt[4]{e_1-e_2}\,.\sqrt{s-e_2}+\sqrt[4]{e_1-e_2}\,.\sqrt{s-e_3}}=lt,$$

$$L_0=1,$$

$$L_1=1+\left(\frac{1}{2}\right)^2 l^4,$$

$$L_2=1+\left(\frac{1}{2}\right)^2 l^4+\left(\frac{1.3}{2.4}\right)^2 l^8,$$

. . . .

. . . .

$$L=1+\left(\frac{1}{2}\right)^2 l^4+\left(\frac{1.3}{2.4}\right)^2 l^8+\left(\frac{1.3.5}{2.4.6}\right)^2 l^{12}+\ldots$$

when the equation

$$u=\frac{2}{(\sqrt[4]{e_1-e_3}+\sqrt[4]{e_1-e_2})^2}\int_t^1\frac{dt}{\sqrt{(1-t^2)(1-l^4t^2)}}$$

$$=\frac{2}{(\sqrt[4]{e_1-e_3}+\sqrt[4]{e_1-e_2})^2}\left\{L\,.\,\frac{1}{i}\text{ log. nat. }(t+i\sqrt{1-t^2})+\sqrt{1-t^2}\left[\frac{L-L_0}{l}(lt)\right.\right.$$

$$\left.\left.+\frac{2}{3}\,\frac{L-L_1}{l^3}(lt)^3+\frac{2.4}{3.5}\cdot\frac{L-L_2}{l^5}(lt)^5+\ldots\right]\right\},$$

determines such a value for u as satisfies the equation $\wp u=s$, when to $\sqrt{1-t^2}$ is given either one of its two values, and to the log. nat. any one of its infinite values."

It is to be hoped that notwithstanding a few gaps in the demonstrations, this sketch has been elaborate enough to give mathematical students a clear idea of these theories in themselves and in relation to the older nomenclature of Jacobi.